The Haar Measure Existence and Uniqueness $\pm\varepsilon$

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1 Preliminaries

In this section, we discuss essential theorems and concepts from measure theory and topology required to understand the proof of the Haar measure's existence and uniqueness (up to a constant). These definitions and theorems are detailed in Rudin's "Real and Complex Analysis"[Rud87] and Munkres's "Topology."[Mun00]

1.1 Definitions in Measure Theory and Topology

1. **Sigma Algebra (σ-algebra):** A σ-algebra on a set X is a collection $\mathcal{S}$ of subsets of X that satisfies:

   i. $X \in \mathcal{S}$.

   ii. If $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$.

   iii. If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of sets in $\mathcal{S}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.

2. **Measure:** Let X be a set equipped with a σ-algebra $\mathcal{S}$. A measure on $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0, \infty]$ such that

   i. $\mu(\emptyset) = 0$,

   ii. if $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$.

3. **Topology:** A topology on a set X is a collection $\mathcal{T}$ of subsets of X called open sets, satisfying the following properties:

   i. $\emptyset$ and X itself are in $\mathcal{T}$.

   ii. The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

   iii. The intersection of any finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

4. **Borel Sigma Algebra:** The Borel σ-algebra on a topological space X is the smallest σ-algebra containing all the open sets. We denote by $\mathcal{B}(X)$.

5. **Topological Basis** A collection $\mathcal{B}$ of open sets in a topological space X is called a *basis* for the topology if every open set in X is a union of sets in $\mathcal{B}$.

6. **Product Spaces (Uncountable):** Given a family $\{X_\alpha\}_{\alpha\in A}$ of spaces (where A is an index set, possibly uncountable), the product space $\prod_{\alpha\in A} X_\alpha$ is the set of all functions f such that $f(\alpha) \in X_\alpha$ for each $\alpha \in A$. The usual topology on this product space is the product topology, which has as a basis sets of the form $\prod_{\alpha\in A} U_\alpha$, where U_α is open in X_α and $U_\alpha \neq X_\alpha$ for only finitely many α.

7. **Locally Compact Group:** A topological group G is called locally compact if every point $x \in G$ has a compact neighborhood, i.e., there exists an open set U and a compact set K such that $x \in U \subseteq K$. Additionally, the group operations (multiplication and taking inverses) are required to be continuous, and its topology must be Hausdorff.

8. **Outer Regular Measure:** A measure μ on a measurable space $(X, \mathcal{S})$ is said to be outer regular if for every measurable set $A \subseteq X$, the measure of A can be approximated from above by open sets, i.e.,

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}.$$

9. **Inner Regular Measure:** A measure μ on a measurable space $(X, \mathcal{S})$ is said to be inner regular if for every measurable set $A \subseteq X$, the measure of A can be approximated from below by compact subsets, i.e.,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ is compact}\}.$$

10. **Radon Measure:** A Radon measure μ on a topological space X is a measure that is both outer regular and inner regular, and additionally, $\mu(K) < \infty$ for every compact set $K \subseteq X$.

11. **Left Haar Measure:** A left Haar measure on a locally compact group G is a non-zero left-invariant ($\mu(gA) = \mu(A)$) Radon measure μ on G.

1.2 Theorems in Measure Theory and Topology

1. **Riesz Representation Theorem:** Let X be a locally compact Hausdorff space and let $C_c(X)$ be the space of continuous functions with compact support on X. For every positive linear functional Λ on $C_c(X)$

[$\Lambda(f) \geq 0$ for $f \geq 0$] , there exists a unique Radon measure μ on X such that for all $f \in C_c(X)$,

$$\Lambda(f) = \int_X f \, d\mu.$$

2. **Urysohn's Lemma:** Let X be a locally compact Hausdorff space, let K be a nonempty compact subset of X, and U an open subset of X such that $K \subseteq U$. Then there is a function $f \in C_c^+(X)$ such that $f(x) = 1$ when $x \in K$, $\operatorname{supp}(f) \subseteq U$, and $0 \leq f(x) \leq 1$ for all $x \in X$.

3. **Tychonoff's Theorem:** The product of any collection of compact topological spaces is compact in the product topology. That is, if $\{X_\alpha\}_{\alpha \in A}$ is a family of compact spaces, then the product space $\prod_{\alpha \in A} X_\alpha$ equipped with the product topology is compact.

4. **Finite Family property**

   (a) A family $\{F_\alpha\}_{\alpha \in A}$ of subsets of X is said to have the *finite intersection property* if $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ for all finite $B \subseteq A$.

   (b) A topological space X is compact if and only if for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.

2 Motivating Examples and Lie groups

Although locally compact groups most frequently arise in practice as Lie groups, it is not accurate to say that all interesting locally compact groups are Lie groups. To understand this, consider the following two examples:

1. $\prod(\mathbb{Z}/2\mathbb{Z})$: This is the state space of the most basic stochastic process - An infinite sequence of coin tosses.

2. $\mathbb{Q}_p$: These are the p-adic numbers, which are another way to complete the rational numbers.

In the context of Lie groups, proving the existence of a Haar measure is relatively straightforward due to their nature as differentiable manifolds, which allows for the application of differential geometry tools. For a detailed explanation and proofs, refer to the document cited as [Dow24], which also includes additional interesting results.

3 Books

I compiled these notes from numerous sources, primarily drawing from texts that often overlap in content. While no single book was followed, the core material is based on works by Hewitt and Ross[HR63], Folland[Fol95], Deitmar and Echterhoff[DE09], and Ramakrishnan and Valenza[RV99]. Additional references are cited as necessary.

4 Topological groups

Recall that a *topological group* is a group G, together with a topology on the set G such that the group multiplication

$$\begin{aligned} G \times G &\longrightarrow G \\ (x, y) &\longmapsto xy, \end{aligned}$$

and inversion

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto x^{-1}, \end{aligned}$$

are both continuous maps.

Remark 1. *It suffices to require the map $\alpha : (x, y) \mapsto x^{-1}y$ to be continuous.*

We now present some examples of topological groups. The verification that these examples satisfy the defining properties of topological groups is left as an exercise.

Example 1. *Any given group becomes a topological group when equipped with the* discrete topology*, i.e., the topology, in which every subset is open. In this case, we speak of a* discrete group

Example 2. *The additive and multiplicative groups $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \times)$ of the field of real numbers are topological groups with their usual topologies. So is the group $GL_n(\mathbb{R})$ of all real invertible $n \times n$ matrices, which inherits the $\mathbb{R}^{n^2}$-topology from the inclusion $GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, where $M_n(\mathbb{R})$ denotes the space of all $n \times n$ matrices over the reals.*

4.1 Basic theory of topological groups

In this section, we introduce some of the basic theory of topological groups. This section contains the essential theorems needed to understand the proofs of the existence and uniqueness of the Haar measure.

Theorem 2. *Let G be a topological group.*

1. *The topology of G is invariant under translations and inversion; that is, if U is open then so are xU, Ux, and U^{-1} for any $x \in G$. Moreover, if U is open then so are AU and UA for any $A \subseteq G$.*

2. *For every neighborhood U of 1 there is a symmetric neighborhood V of 1 such that $VV \subseteq U$.*

Proof. 1. The first assertion is equivalent to the separate continuity of the map $(x, y) \mapsto xy$ and the continuity of the map $x \mapsto x^{-1}$. The second one follows since $AU = \bigcup_{x\in A} xU$ and $UA = \bigcup_{x\in A} Ux$.

2. Continuity of $(x, y) \mapsto xy$ at 1 means that for every neighborhood U of 1 there are neighborhoods W_1, W_2 of 1 with $W_1W_2 \subseteq U$. The desired set V can be taken to be $W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}$.

□

Theorem 2 effectively gives us two important characterizations of the defining properties of topological groups. Next, we consider functions from the group to the complex plane, and define the basic objects needed for our purposes:

4.2 Translates and Continuity

Definition 3. *Let $f : G \to \mathbb{C}$, and let $y \in G$ then we define the left and right translate of the function f, respectively, as follows:*

$$(L_y f)(x) = f(y^{-1}x)$$

and

$$(R_y f)(x) = f(xy)$$

Remark 4. *We defined left translates as we did so that $L_{xy} = L_xL_y$*

Using the definitions previously outlined 3 , we can formulate a definition of uniform continuity within a group context. Given that the group may not be abelian, it is necessary to differentiate between left and right uniform continuity.

Definition 5. *We say a function f is Left uniform continuous if*

$$\|L_y f - f\|_\infty \to 0$$

and similarly, f is Right uniform-continuous if

$$\|R_y f - f\|_\infty \to 0$$

as $y \to \{1\}$

Remark 6. *Note that above $\|\cdot\|_\infty$ denotes the supremum norm. For the remaining of this paper $\|\cdot\| = \|\cdot\|_\infty$ unless otherwise stated*

We now present a generalization of the well-known theorem from Analysis 1: continuous functions on compact sets are uniformly continuous.

Proposition 7. *If $f \in C_c(G)$ then f is left and right uniformly continuous.*

Proof. We give the proof for $R_y f$; the argument for $L_y f$ is similar. Given $f \in C_c(G)$ and $\epsilon > 0$, let $K = \operatorname{supp}(f)$. For every $x \in K$ there is a neighborhood U_x of 1 such that $|f(xy) - f(x)| < \frac{\epsilon}{2}$ for $y \in U_x$, and there is a symmetric neighborhood V_x of 1 such that $V_x V_x \subseteq U_x$. The sets xV_x ($x \in K$) cover K, so there exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^n x_j V_{x_j}$. Let $V = \bigcap_{j=1}^n V_{x_j}$; we claim that $\|R_y f - f\| < \epsilon$ for $y \in V$.

If $x \in K$ then there is some j for which $x_j^{-1} x \in V_{x_j}$, this is because $K \subseteq \bigcup_{j=1}^n x_j V_{x_j}$. Therefore, $xy = x_j(x_j^{-1}xy) \in x_j U_{x_j}$. But then

$$|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Similarly, if $xy \in K$ then $|f(xy) - f(x)| < \epsilon$. But if x and xy are not in K then $f(x) = f(xy) = 0$, so we are done. $\square$

5 Existence

5.1 History and Motivation

The Haar measure, first introduced by Haar in 1933 [Haa33] , established the existence of such a measure on every locally compact group with a countable basis. Weil [Wei65] later extended this to all locally compact groups through the axiom of choice. Alternative proofs that do not rely on this axiom were provided by Cartan[Car40] and Bredon[Bre63], with Alfsen[Alf63] offering further simplifications. Furthermore, these results were used by von Neumann [vN33] to establish Hilbert's fifth problem in the case of compact groups.

5.2 What if we had a measure?

Let us first begin with a definition:

Definition 8 ($C_c^+(G)$ - space)**.**

$$C_c^+(G) = \{f \in C_c(G) : f \geq 0, \|f\|_\infty > 0\}$$

In other words, this is the space of compactly supported functions on G *that are not identically* 0

To clarify the rationale behind the proof's methodology, we will start by presenting a proposition that sheds light on the steps taken to establish the existence of the Haar Measure.

Proposition 9. *Let* μ *be a Radon measure on the locally compact group* G*, then* μ *is a left Haar measure if and only if*

$$\int_G L_y f \, d\mu = \int_G f \, d\mu$$

for every $f \in C_c^+(G)$ *and every* $y \in G$*.*

Proof. $\Longrightarrow$ The forward direction is effectively immediate via an approximation by simple measurable functions

$\Longleftarrow$ Define the positive linear operator by

$$I(f) = \int_G f d\mu$$

because this is a positive linear functional the Riez Representation states that the measure μ is unique. Therefore, because

$$\begin{aligned} I(L_y f) &= \int_G f(y^{-1}x)d\mu(x) \\ &= \int_G f(x)d\mu_y(x) \qquad \text{where } \mu_y(A) = \mu(yA) \end{aligned}$$

and by the assumption $I(f) = I(L_y f)$ we have that

$$\int_G f d\mu = \int_G f d\mu_y$$

thus by the uniqueness of the measure μ we conclude that $\mu = \mu_y$ which concludes this proof

□

5.2.1 Quick Intuition behind the proof

The essence of Proposition 9 is that demonstrating the existence of a left Haar measure simply requires identifying a left-invariant linear functional on $C_c(G)$. To achieve this, we aim to construct a device that is invariant under left translations to compare functions within $C_c^+(G)$. To elucidate how one might go about doing something like this let us first give some intuition [2]:

1. Imagine a function $\phi \in C_c^+(G)$ that is bounded by 1, equals 1 on a small open set, and is supported in a barely larger open set U. [In the $G = \mathbb{R}$ case you can think of ϕ as a rectangle]
2. If $f \in C_c^+(G)$ is sufficiently slowly varying so that it is essentially constant on the left translates of U, f can be well approximated by a linear combination of left translates of ϕ: $f \approx \sum c_j L_{x_j}\phi$[You can think of this as an approximation of f by rectangles]
3. If μ were a left Haar measure on G, we would then have $\int f\, d\mu \approx (\sum c_j) \int \phi\, d\mu$. This approximation will get better and better as the support of ϕ shrinks to a point [You can think of this as a Riemann sum]

[2] In the previous iteration of this document I had a different way to give intuition. This is a slightly different way which is based on Riemann Integration

4. If we, somehow, can normalize so that we can cancel out the factor of $\int \phi \, d\mu$ on the right then we will obtain $\int f \, d\mu$ as a limit of the sums $\sum c_j$.

In the following sections, we make this more precise.

5.3 Existence Proof

5.3.1 Step 1 - Haar Covering Number

Definition 10 (Haar Covering Number)**.** *Let $f, \phi \in C_c^+(G)$ and define $L_x f(y) = f(xy)$, then*

$$(f : \phi) = \inf \left\{ \sum_{i=1}^{n} c_i : f \leq \sum_{i=1}^{n} c_i L_{x_i} \phi \text{ for some } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in G \right\}$$

When objects are defined in this manner, one should immediately ask *Could the set over which we optimize be empty? Could it be unbounded?* The answer to these questions is *No*, and this is a consequence of the following proposition.

Proposition 11. *Let $f, \phi \in C_c^+(G)$, then $0 < (f : \phi) < \infty$*

Proof. We will in fact prove a stronger statement i.e

$$\frac{\|f\|}{\|\phi\|} \leq (f : \phi) \leq C(f, \phi) \frac{\|f\|}{\|\phi\|}$$

where $C(f, \phi)$ is a constant depending on both functions. Consider the set $U_\epsilon = \{\phi > \frac{1}{2}\|\phi\|\}$, it is clear that this set is non-empty as ϕ is continuous and compactly supported so its supremum is achieved. Moreover by continuity of ϕ we obtain that this set is open and therefore by compactness of $\mathrm{supp}(f)$, there exits M translates of U covering $\mathrm{supp}(f)$. That is we have that, $\mathrm{supp}(f) \subseteq \cup_{i=1}^{M} g_i U_\epsilon$ where $g_1, \dots, g_M$ are in G, and therefore, their inverses $g_1^{-1}, \dots, g_M^{-1}$ are also in G. Let us take precisely these inverses $g_1^{-1}, \dots, g_M^{-1}$ to be the translates ("$x_1, \dots, x_n$") theorem 10. Let $x \in \mathrm{supp}(f)$, then without loss of generality say that $x = g_1 x_\epsilon$ for $x_\epsilon \in U_\epsilon$. Then we have that:

$$\begin{aligned}\frac{2\sum_l L_{g_l^{-1}}\phi(x)}{\|\phi\|} &\geq \frac{2L_{g_1^{-1}}\phi(x)}{\|\phi\|} \\ &\geq \frac{2\phi(x_\epsilon)}{\|\phi\|} \\ &\geq 1\end{aligned}$$

where the first inequality holds as $\phi \in C_c^+(G)$, Therefore:

$$f(x) \leq \|f\| \leq 2\frac{\|f\|}{\|\phi\|}\sum_{j=1}^{m} L_{g_j^{-1}}\phi(x)$$

Thus $(f:\phi) \leq 2M\frac{\|f\|}{\|\phi\|}$. This establishes the desired upper bound. Next, we give a proof of the lower bound. Let $x_1,\ldots,x_n$ satisfying $f \leq \sum_{i=1}^{n} c_i L_{x_i}\phi$. Because $f \in C_c^+$ we have that there exits some x^* such that $\|f\| = f(x^*)$ then

$$\|f\| = f(x^*) \leq \sum_i c_i L_{x_i}\phi(x^*) \leq \|\phi\|\sum_i c_i$$

thus

$$\frac{\|f\|}{\|\phi\|} \leq \sum_i c_i$$

as this works for any possible weights $c_1,\ldots,c_m$, this establishes the claim. □

Lemma 12 (Properties of the Haar covering number)**.** *Suppose that $f, g, \phi \in C_c^+$.*

(a) $(f:\phi) = (L_x f:\phi)$ *for any* $x \in G$.

(b) $(cf:\phi) = c(f:\phi)$ *for any* $c > 0$.

(c) $(f+g:\phi) \leq (f:\phi) + (g:\phi)$.

(d) $(f:\phi) \leq (f:g)(g:\phi)$.

Note that (a) tells us that the Haar covering number is left G-invariant. Then (b) and (c) tell us that it is sub-linear when the "relative" function ϕ (the second argument) is fixed. We refer to property (d) as the *"Tower Property"*

Proof. Properties (a),(b),(c) follow quickly from the definition thus these are left as exercises. (d) is a little less clear but it follows immediately from the observation that if

$$f \leq \sum_j c_j L_{x_j} g$$

and

$$g \leq \sum_k d_k L_{y_k} \phi$$

then

$$f \leq \sum_{j,k} c_j d_k L_{x_j y_k} \phi.$$

Thus because

$$\sum_{j,k} c_j d_k = \left(\sum_j c_j \right) \left(\sum_k d_k \right)$$

(d) follows. $\square$

5.3.2 Step 2 - Almost Linear functional

Following Haar's original idea, we make a normalization by choosing, a *fixed*, $f_0 \in C_c^+$, and defining:

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)} \quad \text{for } f, \phi \in C_c^+.$$

From Lemma 12 we obtain that for each fixed ϕ the functional I_ϕ is left-invariant and it is sub-linear. Moreover the tower property gives us the following:

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

We refer to these as "tower bounds". We show that I_ϕ exhibits approximate additivity for small supp(ϕ). This approximation will enable us to estimate a positive linear functional, upon which we can employ the Riesz Representation Theorem to obtain the desired Haar measure.

Lemma 13. *If $f_1, f_2, \phi \in C_c^+$ and $\epsilon > 0$, then there exists a neighborhood V of 1 such that*

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon$$

whenever supp$(\phi) \subseteq V$.

Proof. Fix $g \in C_c^+(G)$ such that $g = 1$ on supp$(f_1 + f_2)$ and let δ be a positive number (to be determined later). Let $h = f_1 + f_2 + \delta g$, and put

$$h_i = \frac{f_i}{h} \qquad\qquad \text{for } i = 1, 2$$

with $h_i = 0$ wherever $f_i = 0$ (this is to rule out a 0/0 case). Then $h_i \in C_c^+(G)$, so by Proposition 7 there is a neighborhood V of 1 in G such that $|h_i(x) - h_i(y)| < \delta$ for $i = 1, 2$ and $y^{-1}x \in V$. Suppose $\phi \in C_c^+(G)$ and supp$(\phi) \subseteq V$. If $h \leq \sum c_j L_{x_j} \phi$ then

$$f_i(x) = h(x)h_i(x) \leq \sum c_j \phi(x_j^{-1}x)h_i(x) \leq \sum c_j \phi(x_j^{-1}x)[h_i(x_j) + \delta],$$

because $|h_i(x) - h_i(x_j)| < \delta$ whenever $x_j^{-1}x \in \text{supp}(\phi)$. Since $h_1 + h_2 \leq 1$, this gives

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum c_j[h_1(x_j) + \delta] + \sum c_j[h_2(x_j) + \delta] \leq \sum c_j[1 + 2\delta].$$

Then taking the infimum of all such sums $\sum c_j$, and using the sub-additivity of I_ϕ we obtain:

$$I_\phi(f_1) + I_\phi(f_2) \leq (1 + 2\delta)I_\phi(h) \leq (1 + 2\delta)I_\phi(f_1 + f_2) + \delta I_\phi(g).$$

By the tower property, we can reach the desired conclusion by taking δ small enough so that

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \epsilon.$$

$\square$

5.3.3 Step 3 - Finding our linear functional

We now possess the necessary tools to state and prove the existence of the Haar measure. The central concept of this proof involves constructing a positive linear functional I as the limit of our nearly positive linear functions I_ϕ within an appropriate space.

Theorem 14. *Every locally compact group G possesses a left Haar Measure*

Proof. We will use Tychonoff's theorem and the Riesz Representation Theorem in this proof. For each $f \in C_c^+$ let X_f be the interval $[(f_0 : f)^{-1}, (f : f_0)]$, and let $X = \prod_{f \in C_c^+} X_f$. Then X is compact by Tychonoff's theorem, and by the tower bounds, every $I_\phi \in X$ (in the sense that $I_\phi(f) \in X$ for all f). For each open neighborhood V of 1, let:

$$K(V) = \mathrm{Cl}_X(\{I_\phi : \mathrm{supp}(\phi) \subseteq V\})$$

Where $\mathrm{Cl}_X(A)$ denotes the closure of a set A in X. It is clear that $K(\bigcap_n V_j) \subseteq \bigcap_n K(V_j)$. Similarly, by definition

$$K\left(\bigcap_{j=1}^{n} V_j\right) = \mathrm{Cl}_X\left(\{I_\phi : \mathrm{supp}(\phi) \subseteq \bigcap_{j=1}^{n} V_j\}\right)$$

is non-empty by Urysohn's Lemma. Therefore, because X is compact,the Finite Family intersection property 4, there exits I_0 such that

$$I_0 \in \bigcap_{V \ni 1} K(V)$$

left-invariant positive linear functional on $C_c(G)$. Note that I, which lies in a product of closed intervals excluding zero, cannot be the zero function on $C_c(G)$, so that the extended functional will likewise be nontrivial.

Since I is in the intersection of the closure of the sets $\{I_\phi : \mathrm{supp}(\phi) \subseteq U\}$, it follows that every open neighborhood of I in the product X intersects each of the sets $\{I_\phi : \mathrm{supp}(\phi) = U\}$. We may unwind this assertion as follows:

For every open neighborhood U of 1, and for every trio of functions $f_1, f_2, f_3 \in C_c^+$ and every $\varepsilon > 0$, there exists a function $\phi \in C_c^+$ with $\mathrm{supp}(\phi) \subseteq U$ such that $|I(f_j) - I_\phi(f_j)| < \varepsilon, j = 1, 2, 3$.

This statement clearly extends to any finite collection of f_j by the definition of product topology. Thus, given $f \in C_c$ and $a \in \mathbb{R}$, we may simultaneously make $I(af)$ arbitrarily close to $I_\phi(aI)$ and $aI(f)$ arbitrarily close to $aI_\phi(f)$. Using Lemma 12, this shows that $I(cf) = cI(f)$. Similarly, we have that I is left translation-invariant and at least subadditive. To see that I is in fact additive, we use Lemma 13 to choose a neighborhood U of 1 such that

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \frac{\varepsilon}{4}$$

whenever $\text{supp}(\phi) \subseteq U$. Then choose ϕ with $\text{supp}(\phi) \subseteq U$ such that $I(f_1), I(f_2)$, and $I(f_1 + f_2)$ all likewise lie within $\varepsilon/4$ of $I_\phi(f_1), I_\phi(f_2)$, and $I_\phi(f_1 + f_2)$, respectively. Since ε is arbitrary, it follows at once from the inequality above and the general sublinearity of I, that $I(f_1 + f_2) = I(f_1) + I(f_2)$, as required.

Finally, extend I to a positive left translation-invariant linear functional on $\mathcal{C}_c(G)$ by setting $I(f) = I(f^+) - I(f^-)$. As we remarked above, in view of our discussion of translation-invariant radon measures 9 and the Riesz representation theorem, this implies that G admits a left Haar measure μ and completes the existence proof.

□

6 Uniqueness

Theorem 15. *If λ and μ are left Haar measures on G, there exists $c \in (0, \infty)$ such that $\mu = c\lambda$.*

Proof. It is clear that $\mu = c\lambda$ is equivalent to the claim that the ratio

$$\frac{\int f d\lambda}{\int f d\mu}$$

is the same for all $f \in C_c^+(G)$. Suppose then that $f, g \in C_c^+(G)$. Fix a symmetric compact neighborhood V_0 of 1 and set

$$A = (\text{supp } f)V_0 \cup V_0(\text{supp } f), \quad B = (\text{supp } g)V_0 \cup V_0(\text{supp } g).$$

Then A and B are compact, and for $y \in V_0$, $f(xy) - f(x)$ and $g(yx) - g(x)$ are supported in A and B respectively, as functions of x.

Given $\varepsilon > 0$, by Proposition 7 there is a symmetric neighborhood $V \subseteq V_0$ of 1 such that $|f(xy) - f(x)| < \varepsilon$ and $|g(yx) - g(x)| < \varepsilon$ for all x when $y \in V$. Pick $h \in C_c^+(G)$ with $h(x) = h(x^{-1})$ and $\text{supp}(h) \subseteq V$. Then

$$\int h d\mu \int f d\lambda = \iint h(y) f(x) d\lambda(x) d\mu(y) = \iint h(y) f(yx) d\lambda(x) d\mu(y),$$

and since $h(x) = h(x^{-1})$,

$$\begin{aligned}
\int h d\lambda \int f d\mu &= \iint h(x) f(y) d\lambda(x) d\mu(y) \\
&= \iint h(y^{-1}x) f(y) d\lambda(x) d\mu(y) \\
&= \iint h(x^{-1}y) f(y) d\mu(y) d\lambda(x) \\
&= \iint h(y) f(xy) d\mu(y) d\lambda(x). \\
&= \iint h(y) f(xy) d\lambda(x) d\mu(y).
\end{aligned}$$

(Fubini's theorem is applicable since all the integrals are effectively over sets that are compact and hence of finite measure.) Therefore,

$$\begin{aligned}
\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| &= \left| \iint h(y) f(xy) - f(yx) d\lambda(x) d\mu(y) \right| \\
&\leq \varepsilon \lambda(A) \int h d\mu.
\end{aligned}$$

In the same way,

$$\left| \int h d\lambda \int g d\mu - \int h d\mu \int g d\lambda \right| \leq \varepsilon \lambda(B) \int h d\mu.$$

Dividing these inequalities by $\int h d\mu \int f d\mu$ and $\int h d\mu \int g d\mu$, respectively, and adding them, we obtain

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| \leq \varepsilon \left[\frac{\lambda(A)}{\int f d\mu} + \frac{\lambda(B)}{\int g d\mu} \right].$$

Note that these rations are well-defined since the functions f and g are both in $C_c^+(G)$. Since ε is arbitrary, the ratio of the integrals with respect to λ and μ is the same for f and g, which is what we needed to show.

$\square$

7 Haar Measure on $\mathbb{Q}_p$

7.1 Some basics of $\mathbb{Q}_p$ p-adic numbers

Fix p prime then, by the Fundamental Theorem of Arithmetic, any $r \in \mathbb{Q}$ can be written as

$$r = p^m q$$

for $m \in \mathbb{Z}$ and q is a rational number whose numerator and denominator are not divisible by p

Definition 16. *Let p be a fixed prime number, then for $r \in \mathbb{Q}$*

$$|r|_p = \frac{1}{p^m} = p^{-m} \qquad \text{for } r \neq 0$$
$$|0|_p = 0.$$

where m satisfies the condition above.

Example 3. *Consider*

$$\frac{140}{297} = 2^2 \times 3^{-3} \times 5 \times 7 \times 11^{-1}$$

then

$$\left|\frac{140}{297}\right|_2 = \frac{1}{4}, \quad \left|\frac{140}{297}\right|_{13} = 1$$
$$\left|\frac{140}{297}\right|_3 = 3^3, \quad \left|\frac{140}{297}\right|_5 = \frac{1}{5}$$

7.2 Properties

1. $|x|_p \geq 0$ for all $x\ in \mathbb{Q}$
2. $|x|_p = 0 \iff x = 0$
3. $|xy|_p = |x|_p |y|_p$ for all $x, y \in \mathbb{Q}$
4. $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for all $\mathbb{Q}$

Note that the properties given above establish that $|\cdot|_p$ is an absolute value, in fact, it is not a norm as it is not homogeneous. However, this is all we need to define a metric on $\mathbb{Q}$, we can do this in the usual way:

$$d_p(r_1, r_2) = |r_1 - r_2|_p$$

Considering this metric, we define the p-adic numbers as the completion of the rational numbers under the p-adic norm. This process mirrors the construction of real numbers, which are the completion of the rationals under the Euclidean norm. In class, we explored another characterization of the p-adic numbers; both characterizations are equivalent, as the subsequent theorem precisely states.

Theorem 17. *If $m \in \mathbb{Z}$ and $c_j \in \{0, 1, \ldots, p-1\}$ for $j \geq m$, the series*

$$\sum_{j=m}^{\infty} c_j p^j$$

converges in $\mathbb{Q}_p$. Moreover, every p-adic number is the sum of such a series.

A proof of this can be found in section 3 of the book [Gou97]

7.3 Constructing the Haar Measure of $\mathbb{Q}_p$

The title of this section may be misleading as it suggests constructing a measure on $\mathbb{Q}_p$, which is not my aim. Instead, I will present the measure in a manner akin to the Lebesgue measure on the real line. The majority of the material here is derived from [Des22], where detailed discussions can be found. To avoid redundancy, I will not delve into full details but rather provide an overview.

We begin by considering the closed ball in the space and looking at some of its properties. For $r \geq 0$ and $x \in \mathbb{Q}_p$ consider the closed ball

$$\overline{B}(x, r) = \{y \in \mathbb{Q}_p : |x - y|_p \leq r\}$$

There are some interesting properties of this object that will help us build the Left Haar measure of $\mathbb{Q}_p$. We first note that because $|\cdot|_p$ takes only on values of the form p^k for k an integer, it follows that for all $r > 0$, there exits some $\epsilon > 0$ such that

$$\overline{B}(x, r) = B(x, r + \epsilon)$$

In particular this implies that the balls are closed and open.

Remark 18. *It follows that $\mathbb{Q}_p$ is a Cantor set, that is, totally disconnected but has no isolated points*

It is clear that every point within a ball is a center, and if two balls intersect, one must be contained within the other. This significant observation is noted as a remark.

Remark 19. *If two balls intersect then one ball is contained in the other.*

Moreover, because we have defined $\mathbb{Q}_p$ as the completion of the rationals under the p-adic norm then it follows that $\mathbb{Q}_p$ is a separable metric space, which leads us to our next remark

Remark 20. *$\mathbb{Q}_p$ is a separable metric space $\implies$ Every open set is the union of balls*

Let $\mathbb{Z}_p = \overline{B}(0,1)$, and let μ be a left Haar measure on $\mathbb{Q}_p$ then we may normalize to make the measure of the p-adic integers $\mathbb{Z}_p$ be of measure 1, that is define a new left Haar measure λ as

$$\lambda(E) = \frac{1}{\mu(\mathbb{Z}_p)}\mu(E)$$

Next, we will make a series of observations that will lead to the desired formula for the Left Haar measure on $\mathbb{Q}_p$:

1. By translation invariance we have that $\lambda(\overline{B}(x,1)) = 1$ for all $x \in \mathbb{Q}_p$
2. If $m > 0$, then $\overline{B}(x,p^m) = \bigsqcup_{i=1}^{p^m} \overline{B}(x_i,1)$, for some $\{x_i\}_i$.
3. Every open set U can be expressed as

   $$U = \bigsqcup_{i=1}^{\infty} \overline{B}(x_i, p^{m_i})$$

   for some $\{x_i\}_i$, note that we may take these balls to be disjoint by Remark 20

Combining all of these observations we obtain that:

$$\lambda(U) = \sum_{i=1}^{\infty} p^{m_i}$$

therefore by outer regularity we conclude that

$$\lambda(E) = \inf\left\{\sum_{j=1}^{\infty} p^{m_j} : E \subseteq \bigcup_{j=1}^{\infty} B(p^{m_j}, x_j)\right\}.$$

this is the representation of this Haar measure I wanted to present here, personally, I think it is quite beautiful how the "form" of the outer Lebesgue measure seems to be the same in both these ways of completing of the rationals.

8 Unimodularity

In this section, we explain unimodularity, a concept that shows how the right and left Haar measures differ in a group. While abelian groups have identical left and right Haar measures, this is not always the case for non-abelian, locally compact topological groups. Let us consider the following example from [Bum13] :

Example 4. *Consider the group*

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}, y > 0 \right\},$$

then it is easy to see (using change of variables) that the left- and right-invariant measures are:

$$d\mu_L = y^{-2}\, dx\, dy, \quad d\mu_R = y^{-1}\, dx\, dy.$$

respectively, thus they are not the same. However, there are many cases where they do coincide.

Definition 21. *A locally compact group for which every left Haar measure is also a right Haar measure is called a unimodular group*

8.1 The modular function

Definition 22. *Let G be a locally-compact group, and let μ be a Haar measure on G. For $x \in G$ the measure μ_x, defined by $\mu_x(A) = \mu(Ax)$, is a Haar measure again, as for $y \in G$ one has $\mu_x(yA) = \mu(yAx) = \mu(Ax) = \mu_x(A)$.*

Therefore, by the uniqueness of the Haar measure, there exists a number $\Delta(x) > 0$ with $\mu_x = \Delta(x)\mu$. In this way one gets a map $\Delta : G \to \mathbb{R}_{>0}$, which is called the modular function of the group G. If $\Delta = 1$, then G is unimodular.

Intuitively, the modular function does exactly what we want it to do: measure how far away we are from right-invariance. To see this, observe that μ_x is a right shift of μ, hence Δ being identically 1, exactly states that μ is left and right invariant.

We will end this section with the statement and proof of some important properties and tools in the theory of the modular function, but before that we need a lemma:

Lemma 23. *Let ν be a Haar measure on G. Then for every $f \in C_c(G)$ the function $s \mapsto \int_G f(xs)d\nu(x)$ is continuous on G.*

Proof. We have to show that for a given $s_0 \in G$ and given $\varepsilon > 0$ there exists a neighborhood U of s_0 such that for every $s \in U$ one has

$$\left| \int_G f(xs) - f(xs_0)d\nu(x) \right| < \varepsilon$$

. Replacing f by $R_{s_0}f(x) = f(xs_0)$, we are reduced to the case $s_0 = e$. Let K be the support of f, and let V be a compact symmetric unit-neighborhood. For $s \in V$ one has $\mathrm{supp}(R_s f) \subseteq KV$. Let $\varepsilon > 0$. As f is uniformly continuous, there is a symmetric unit-neighborhood W such that for $s \in W$ one has $|f(xs) - f(x)| < \frac{\varepsilon}{\nu(KV)}$. For $s \in U = W \cap V$ one therefore gets

$$\begin{aligned} \left| \int_G f(xs) - f(x)d\nu(x) \right| &\leq \int_{KV} |f(xs) - f(x)|\, d\nu(x) \\ &< \frac{\varepsilon}{\nu(KV)}\nu(KV) = \varepsilon. \end{aligned}$$

□

Next we give the promised results, and some brief arguments. Some of the details are skipped and are left as exercises.

Theorem 24.

1. *The modular function $\Delta : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism.*

2. *$\Delta = 1$ if G is abelian or compact.*

3. *For $y \in G$ and $f \in L^1(G)$ one has $R_y f \in L^1(G)$ and*

$$\int_G R_y f(x)\, dx = \int_G f(xy)\, dx = \Delta(y^{-1}) \int_G f(x)\, dx.$$

4. *The equality*

$$\int_G f(x^{-1})\Delta(x^{-1})\, dx = \int_G f(x)\, dx$$

holds for every integrable function f.

Proof.

(3) Part (3) is clear if f is the characteristic function $\mathbf{1}_A$ of a measurable set A. It follows generally by the usual approximation argument.

(1) We now prove part (a) of the theorem. For $x, y \in G$ and a measurable set $A \subseteq G$, one computes

$$\Delta(xy)\mu(A) = \mu_{xy}(A) = \mu(Axy) = \mu_y(Ax)$$

$$= \Delta(y)\mu(Ax) = \Delta(y)\Delta(x)\mu(A).$$

Choose A with $0 < \mu(A) < \infty$ to get $\Delta(xy) = \Delta(x)\Delta(y)$. Hence Δ is a group homomorphism. Next, we show continuity. Let $f \in C_c(G)$ with $c = \int_G f(x)\, dx \neq 0$. By part (3) we have

$$\Delta(y) = \frac{1}{c} \int_G f(x^{-1})\, dx = \frac{1}{c} \int_G R_{y^{-1}} f(x)\, dx.$$

So the function Δ is continuous in y by the lemma above.

(2) The Abelian case is clear. If G is compact, then so is the image of the continuous map Δ. As Δ is a group homomorphism, the image is also a subgroup of $\mathbb{R}_{>0}$. But the only compact subgroup of the latter is the trivial group $\{1\}$, which means that $\Delta \equiv 1$.

(4) Finally, for part (4) of the theorem let $f \in C_c(G)$ and set $I(f) = \int_G f(x^{-1})\Delta(x^{-1})\,dx$. Then, by part (c),

$$I(L_z f) = \int_G f(z^{-1}x^{-1})\Delta(x^{-1})\,dx = \int_G f((xz)^{-1})\Delta(x^{-1})\,dx$$

$$= \Delta(z^{-1}) \int_G f(x^{-1})\Delta((xz)^{-1})\,dx = \int_G f(x^{-1})\Delta(x^{-1})\,dx = I(f).$$

It follows that I is an invariant integral; hence there is $c > 0$ with $I(f) = c\int_G f(x)\,dx$. To show that $c = 1$ let $\varepsilon > 0$ and choose a symmetric unit-neighborhood V with $|1 - \Delta(s)| < \varepsilon$ for every $s \in V$. Choose a nonzero symmetric function $f \in C_c^+(V)$. Then

$$\begin{aligned} |1-c| \left| \int_G f(x)\,dx \right| &= \left| \int_G f(x)\,dx - I(f) \right| \\ &\leq \left| \int_G |f(x) - f(x^{-1})\Delta(x^{-1})|\,dx \right| \\ &= \left| \int_V f(x)|1 - \Delta(x^{-1})|\,dx \right| \\ &< \varepsilon \left| \int_G f(x)\,dx \right|. \end{aligned}$$

So one gets $|1 - c| < \varepsilon$, and as ε was arbitrary, it follows $c = 1$ as claimed.

□

We finalize these notes, by giving some examples of compact groups:

Example 5. *Some classical easy examples of compact groups and therefore unimodular groups include*

1. $O(n)$

2. $SO(n)$

3. $U(n)$

where the topology comes from identifying their elements with vectors in the corresponding large Euclidean space.

9 References

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